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On nonseparability of Holder-Lipschitz space

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Abstract—Many spaces that we come across in calculus and geometry are separable. For mathematicians separable spaces have some attractive properties arising from the ability to represent every element of a space as a limit of sequence of elements from a countable set; as any real number can be represented as limit of a sequence of rational numbers. The purpose of this work is to show nonseparability of the space $C^{n,\lambda}(\Omega)$ – the space of continuous and n-times continuously differentiable functions with Holder-Lipschitz condition imposed on the last n-th derivative with some λ coefficient.

Index Terms— approximation error, Banach space, convergence rate of approximation, Holder-Lipschitz condition, everywhere dense set, nonseparable space, norm, piecewise linear approximation, space of continuous functions

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1 Introduction

Definition 1.
Set $M \subset B$ is called everywhere dense in a normed space $y \in B$ exists $0 > \varepsilon$ and any arbitrarily small $x \in B$ if for any ,B , such as $\|x-y\|_B < \varepsilon$

Definition 2.
A normed space is called separable if it contains a finite or countable everywhere dense set.

Usually, as an instance of Banach nonseparable space we consider the space $l^{\propto}$(space of limited sequences). It contains noncountable set with pairwise distance between sequences equal to one, as such sequences we can take all sequences of zero and one. The trivial example of a separable space is a set of real numbers, as it includes everywhere dense set of rational numbers. Another illustration of a separable space is the space of continuous functions 0,1[C], since according to Weierstrass approximation theorem the space of polynomials with rational coefficients is a countable everywhere dense subspace of the space 0,1[C].

Theorem 1
.$^{n,\lambda}(\Omega)$ is a Banach space$C$

2 Result and Discussion

Before discussing the question related to separability of the space $C^{n,\lambda}(\Omega)$, we will consider the behavior of piece-wise linear approximations on the functions from this space. We will study one-dimensional case. Consideration of multidimensional case does not differ from the study of one-dimensional version, except abundance of indexes and variables. Moreover we will scrutinize $C^{0,1}[a,b]$ as prove for the general case .$^{0,\lambda}[a,b]$ is analogous$C$

We will start the discussion with the space of continuous functions $C[a,b]$ with the introduced norm on it:

$$\|f\|_C = \sup_{x\in[a,b]} |f(x)|$$

Let us consider a function $f(x) \in C[a,b]$. We will partition the segment $[a,b]$ into n equal parts and consider some piecewise linear approximation f_n. It is a known fact that for $f(x) \in C[a,b]$ occurs that $f_n \to f$ when $n \to \infty$ in a sense of norm defined on $C[a,b]$.
In other words:

$$\lim_{n\to\infty} max_{x\in[a,b]}|f(x) - f_n(x)| = 0$$

In fact, to solve applied problems besides convergence, we need to know a convergence rate of approximation or the number in which a segment is required to be portioned in order to piecewise linear approximation error will not exceed the predetermined value. In case of the space C$[a,b]$ with the introduced norm the answer is unknown.
Nevertheless, if we consider the space $C^{0,1}[a,b]$, we can estimate the piece-wise linear approximation error. The space $C^{0,1}[a,b]$ will be scrutinized with the norm:

$$\|f\|_{C^{0,1}} = \sup_{x\in[a,b]} |f(x)| + \sup_{\substack{x_1,x_2\epsilon[a,b] \\ x_1\neq x_2}} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

Let us consider a function $f(x) \in C^{0,1}[a,b]$, partition of the segment $[a,b]$ into n equal pieces and a corresponding to this partition liner approximation f_n.

$$h = \frac{b-a}{n}$$

The maximum value of piece-wise linear approximation error in a sense of norm defined on $C[a,b]$ is:

$$\|f - f_n\|_C \leq \|f\|_{C^{0,1}} \frac{h}{2} \leq \|f\|_{C^{0,1}} \frac{b-a}{2n}$$

This estimation can be applied in a range of realms.

Now let us evaluate the error in a sense of norm on $C^{0,1}[a,b]$:

$$\|f - f_n\|_{C^{0,1}(\Omega)} \leq \|f\|_{C^{0,1}} \frac{b-a}{2n} +$$

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$$+\sup_{\substack{x_1,x_2\epsilon[a,b]\\ x_1\neq x_2}}\frac{|f(x_1)-f(x_2)-(f_n(x_1)-f_n(x_2))|}{|x_1-x_2|}$$

If in the calculated estimation, the second term tends to zero when $n\to\infty$, then it would be easy to build a countable everywhere dense set to prove separability of the space $C^{0,1}[a,b]$.

Lemma. If in some Banach space B exists an uncountable set $M\subset B$ such that pairwise distances between its elements are bigger than some fixed constant d, i. e $\forall f_1,f_2\in B, f_1\neq f_2$ the inequality is held

$$\|f_1-f_2\|_B\geq d$$

then B is not separable.

Thus, to prove nonseparability of the space $C^{0,1}[a,b]$, it is required to build set M. As such set we can take the set of functions $f_c(x)=|x-c|, c\in(a,b)$. It is obvious that in this case, M is uncountable as the set (a,b) is uncountable. Let us show that the required inequality will be held for this set.

$$\left\|f_{C_1}-f_{C_2}\right\|_{C^{0,1}[a,b]}\geq 2, c_1,c_2\in(a,b), c_1\neq c_2$$

$$\left\|f_{C_1}-f_{C_2}\right\|_{C^{0,1}[a,b]}=\sup_{x\in[a,b]}\left||x-c_1|-|x-c_2|\right|+$$

$$+\sup_{\substack{x_1,x_2\epsilon[a,b]\\ x_1\neq x_2}}\frac{|f(x_1)-f(x_2)-(f_n(x_1)-f_n(x_2))|}{|x_1-x_2|}$$

Let us put $x_1=c_1, x_2=c_2$ in the second term, we will obtain

$$\frac{\left||c_1-c_1|-|c_2-c_1|-(|c_1-c_2|-|c_2-c_2|)\right|}{|c_1-c_2|}=$$

$$=\frac{\left||c_2-c_1|+|c_1-c_2|\right|}{|c_1-c_2|}=\frac{2|c_1-c_2|}{|c_1-c_2|}=2$$

To summarize, in case of the space $C^{0,1}[a,b]$ we proved the following theorem:

Theorem: the space $C^{n,\lambda}(\Omega)$ is not separable.

In particular, nonseparability of the space $C^{0,1}[a,b]$ means that there are functions for which piece-wise linear approximation error does not tend to zero when the partition of the segment $[a,b]$ is milled and $n\to\infty$. Let us build an example of such function $f(x)$ but in the space $C^{0,1}[0,2]$.

Function example.
The main idea is following: we partition the segment [0,2] into two equal parts and put the pick of height 1 on the left side. Then we partition the right side into two equal halves and put the pick of height ½ on the left halve. Further, again, the rest we partition into two equal pieces and put on the left piece the pick of height ¼. We infinitely continue this process.
It is obvious, that $\|f\|_{C^{0,1}}=3$, as the first summand in the defined norm equals one and the second results in two, thus, in sum we obtain three. Let us show that for a piecewise linear approximation $f_n(x)$ built on a uniform partition of the segment into n pieces, the following formula is held:

$$\lim_{n\to\infty}\|f-f_n\|_{C^{0,1}}\neq 0.$$

Full analysis of this limit would be extremely complicated and time-consuming. Therefore, to proof this fact we will simply show that the following claim is false: limit exists and equals zero. So, we have some numerical sequence

$$\{\|f-f_n\|_{C^{0,1}}\}_{n=1}^{\propto}.$$

and as it is known if a sequence converges to some limit then to the same limit converges any of its subsequences. As such subsequence we will take the elements with indexes 2^{n+1} and try to find a limit of the sequence

$$\{\|f-f_{2^{n+1}}\|_{C^{0,1}}\}_{n=1}^{\propto}.$$

This type of subsequence was chosen in order to make boundaries of the segments partition becoming points of local maxima and minima.
Not hard to see that:

$$\lim_{n\to\infty}\|f-f_{2^{n+1}}\|_{C^{0,1}}=2$$

Therefore, it implies that even if

$$\lim_{n\to\infty}\|f-f_n\|_{C^{0,1}}$$

exists, it equals two not zero.

3 Conclusion

A range of theorems can be proved constructively only for separable spaces. If a space is not separable, it becomes more complicated to apply methods of finite elements. To exemplify, we cannot use Fourier series, piece-wise linear approximations and etc. In most cases particularly we deal with separable spaces, as in case of the space $C[0,1]$. However, as it was shown above the space $C^{n,\lambda}(\Omega)$ does not possess the property of separability.

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